Perturbative Renormalization with Flow Equations in Minkowski Space

Georg Keller
Rotherschlössli
CH-6022 Grosswangen

Christoph Kopper, Clemens Schophaus
Institut für Theoretische Physik
Universität Göttingen
Bunsenstraße 9
D-37073 Göttingen

February 7, 2008

Abstract

We show within the Wilson renormalization group framework how the flow equation method can be used to prove the perturbative renormalizability of relativistic massive Φ_4^4. Furthermore we prove the regularity of the renormalized relativistic one-particle irreducible n-point Green functions in the region predicted by axiomatic quantum field theory which ensures that physical renormalization conditions for the two-point function can be imposed.

1 Introduction

The theory of the renormalization group and of effective Lagrangians which was invented by Wilson and his collaborators in 1974 [1] has proved to be a rich and powerful method for many branches of quantum field theory and statistical mechanics. Adapting the Wilson flow equations to perturbation theory Polchinski first applied this method to the renormalization problem of perturbative field theory [2]. Instead of analysing any complicated divergence/convergence properties of the general bare or renormalized Feynman diagram, this access solves the problem of perturbative renormalizability by bounding the solutions of the system of the first order differential flow equations.

In this paper we continue the programme of two of the authors to give mathematically strict proofs of the perturbative renormalizability of any (by naive power counting) renormalizable theory of physical interest using an improved version of Polchinski's method. Namely we show how the flow equation method can be extended to relativistic theories. The first paper in this series modified and improved Polchinski's proof of perturbative renormalizability of Euclidean massive Φ_4^4 [3]. Within the Euclidean framework this improved version of the flow equation method was then applied to a vast range of renormalization problems of perturbative field theory, e.g. the renormalization of composite operators [4], the Zimmermann identities [4], the existence of the short distance expansion [5], Symanzik's improvement programme [6], the construction of the analytical minimal subtraction scheme [7], local Borel summability for massive Φ_4^4 [8] and the renormalization of massless Φ_4^4 [10] and QED [9],[11].

In order to treat the renormalization problem for relativistic theories one has to deal with the fact that in momentum space n-point Green functions can in general only be interpreted as tempered distributions. Therefore renormalization conditions, i.e. the requirement that certain n-point Green functions and some of their derivatives take special values at given points in momentum space which ensures that we are dealing with the right physical constants in the renormalized theory, can only be imposed if supplementary regularity properties can be verified. Restricting to a massive scalar field theory one has in particular to ensure that the renormalized two-point Green function has a pole with residue 1 on the physical mass shell, i.e. the physical mass shell should be in a region of regularity of the renormalized amputated two-point Green function.

In the literature we found two different ways to handle this problem. The first way uses renormalization methods that directly lead to the right renormalization conditions for the renormalized two-point Green function in every order of perturbation theory. Furthermore it is shown that in every order the renormalized n-point Green functions fulfill the LSZ axioms and therefore have the domain of analyticity predicted by axiomatic quantum field theory [12]. Steinmann's method for the renormalization of generalized retarded n-point functions [13] and the renormalization method of Epstein and Glaser for time-ordered operator products [14],[15] solve the problem using this strategy. The other possibility is to renormalize first using some unphysical renormalization scheme like minimal subtraction with analytic or dimensional regularization or to subtract at a point where regularity is evident, e.g. at 0

momentum in a massive theory, and to ensure afterwards that the required conditions can be satisfied in every order of perturbation theory with the help of finite renormalizations [16]. Therefore it is necessary to show that the renormalized n-point functions obtained through the first step have appropriate regions of regularity for the finite renormalizations to be well-defined. The singularity regions of unrenormalized Feynman integrals had been studied in the context of analytical properties of scattering functions in S-matrix theories [17]. Because at that time no method was known that rigorously solved the problems of renormalization, it had to be taken for granted and was widely accepted that renormalization did not change the regularity statements. In 1966 Hepp used the Bogoliubov-Parasiuk subtraction method, which corresponds to imposing renormalization conditions at 0 momentum, to prove the perturbative renormalizability of a relativistic massive scalar field theory [18]. Furthermore he showed that for relativistic massive Φ_4^4 physical renormalization conditions could be obtained by a finite renormalization [18],[19]. He only made a short comment (to our knowledge) on the possibility of transferring the results about singularity surfaces of unrenormalized Feynman integrals to the case of renormalized ones. Chandler proved this to be true for analytically renormalized Feynman integrals in 1970 [20]. Finally Rivasseau pointed out to us the strategy of proving for renormalization in parametric space that the regularity region of the renormalized two-point Green function is sufficiently large to ensure physical renormalization conditions on the mass shell [21].

In this paper we also start from renormalization conditions at 0 momentum similarly as Hepp did and prove the perturbative renormalizability of relativistic massive Φ_4^4. We impose arbitrary conditions at 0 momentum and obtain renormalized n-point functions. (In fact we fix particularly simple renormalization conditions at 0 momentum for simplicity of notation, but the generalization to arbitrary renormalization conditions at 0 momentum can be carried out without difficulties.) Then we show that these renormalized n-point functions have regularity regions that admit physical renormalization conditions, i.e. the renormalization conditions at 0 momentum can be chosen in such a way that physical renormalization conditions on the mass shell are satisfied. In order not to get bothered with the poles of the amputated connected n-point Green functions for partial sums of external momenta lying on the mass shell we analyse the one-particle irreducible Green functions.

The paper is organized as follows: We first derive the flow equations for one-particle irreducible Green functions in the Euclidean theory. Then we show certain analyticity properties of the renormalized Euclidean one-particle irreducible n-point functions for complex momenta. We define the relativistic theory with an ε-regularization due to Speer [22] and prove perturbative renormalizability and the fact that the renormalized one-particle irreducible n-point Green functions become Lorentz invariant tempered distributions in the limit $\varepsilon \to 0$. Using the analyticity properties of the renormalized Euclidean theory we then derive the regularity of the renormalized relativistic one-particle irreducible n-point Green functions in a region that admits physical renormalization conditions for the two-point function.

2 The Flow Equation for regularized Euclidean massive Φ_4^4

In this section we shortly introduce some basic tools of the flow equation method (for details see [3],[4]), which are necessary for the subsequent considerations.

We use the following regularized free Euclidean propagator

$$\tilde{C}_\alpha^{\alpha_0}(p) := \int_{\alpha_0}^{\alpha} d\alpha' \, e^{-\alpha'(p_0^2+\underline{p}^2+m^2)} \quad , \quad 0 < \alpha_0 \le \alpha < \infty \quad . \tag{1}$$

Note that this regularization differs from that used in [3],[4], as it respects analyticity in momentum space. The Fourier transform is denoted as $C_\alpha^{\alpha_0}(x-y)$. The functional Laplace operator $\Delta(\alpha, \alpha_0)$ is defined as

$$\Delta(\alpha,\alpha_0) := \frac{1}{2}\int d^4x \int d^4y \; C_\alpha^{\alpha_0}(x-y)\,\frac{\delta}{\delta\,\Phi(x)}\,\frac{\delta}{\delta\,\Phi(y)} \quad .$$

$\Phi(x)$ may be viewed as an element in $S(\mathbf{R}^4)$. The interaction Lagrangian at scale α_0 is given as a formal power series:

$$L^{\alpha_0,\alpha_0}(\Phi) := \sum_{r\ge 1} g^r \, L_r^{\alpha_0,\alpha_0}(\Phi) \quad .$$

This is the standard Lagrangian including counterterms:

$$L_r^{\alpha_0,\alpha_0}(\Phi) := \int d^4x \, \left(a_r^{\alpha_0}\Phi^2(x) - b_r^{\alpha_0}\Phi(x)\Box\Phi(x) + c_r^{\alpha_0}\Phi^4(x)\right) \quad , \tag{2}$$

where $\Box$ denotes the 4-dim Laplace operator.

The effective Lagrangian

$$L^{\alpha,\alpha_0}(\Phi) := \sum_{r\ge 1} g^r \, L_r^{\alpha,\alpha_0}(\Phi)$$

is defined through

$$e^{\,-L^{\alpha,\alpha_0}(\Phi)\;-I^{\alpha,\alpha_0}} \;:=\; e^{\Delta(\alpha,\alpha_0)}\; e^{-L^{\alpha_0,\alpha_0}(\Phi)} \quad , \tag{3}$$

where I^{α,α_0} collects the terms, which satisfy $\frac{\delta}{\delta\,\Phi} I^{\alpha,\alpha_0} \equiv 0$. (Note that as long as these terms appear we have to keep the volume in (2) finite to be mathematically strict. But as we are only interested in the Φ-dependent terms, we will ignore this point.)

The flow equation for the effective Lagrangian can be obtained by differentiation of (3) with respect to α and is given by

$$\partial_\alpha L^{\alpha,\alpha_0}(\Phi) \,+\, \partial_\alpha I^{\alpha,\alpha_0} \;=\; [\,\partial_\alpha\Delta(\alpha,\alpha_0)\,]\; L^{\alpha,\alpha_0}(\Phi) \tag{4}$$

$$-\frac{1}{2}\int d^4x \int d^4y \left(\frac{\delta}{\delta\,\Phi(x)}\,L^{\alpha,\alpha_0}(\Phi)\right)(\partial_\alpha\,C_\alpha^{\alpha_0}(x-y))\,\frac{\delta}{\delta\,\Phi(y)}\,L^{\alpha,\alpha_0}(\Phi) \quad .$$

Regarding the fields Φ as functions on momentum space, $L_r^{\alpha,\alpha_0}(\Phi)$ can be written as

$$L_r^{\alpha,\alpha_0}(\Phi)=\sum_{n\geq 2}\int\prod_{k=1}^{n-1}\frac{d^4p_k}{(2\pi)^4}\;\Phi(p_k)\,\Phi(-\sum_{j=1}^{n-1}p_j)\;\mathcal{L}_{r,n}^{\alpha,\alpha_0}(p_1,\ldots,p_{n-1}) \quad . \tag{5}$$

$\mathcal{L}_{r,n}^{\alpha,\alpha_0}$ is the r^{th} order contribution to the connected amputated n-point Green function. It enjoys the following properties:

a) $\mathcal{L}_{r,n}^{\alpha,\alpha_0}$ may be assumed symmetric under permutations of $p_1,\ldots,p_n:=-\sum_{j=1}^{n-1}p_j$.

b) $\mathcal{L}_{r,n}^{\alpha,\alpha_0}\equiv 0$ for $n>2r+2$ (connectedness),
$\mathcal{L}_{r,2k+1}^{\alpha,\alpha_0}\equiv 0$ (due to the symmetry $\Phi\to-\Phi$).

c) $\mathcal{L}_{r,n}^{\alpha,\alpha_0}$ is invariant under $O(4)$-transformations of the p_j .

d) $\mathcal{L}_{r,n}^{\alpha,\alpha_0}$ is in $C^\infty([\alpha_0,\infty)\times\mathbf{R}^{4(n-1)})$ as a function of α and $p_1,\ldots,p_{n-1}$.

3 Flow Equations for one-particle irreducible Green functions

3.1 The Generating Functional $\Gamma^{\alpha,\alpha_0}(\Phi_c)$

The generating functional $W_c^{\alpha,\alpha_0}(J)$ of the pertubative, regularized connected Green functions is given by

$$W_c^{\alpha,\alpha_0}(J):=L^{\alpha,\alpha_0}(\Phi)\,|_{\,\Phi=\tilde{C}_\alpha^{\alpha_0}J}+I^{\alpha,\alpha_0}-\frac{1}{2}<J,\tilde{C}_\alpha^{\alpha_0}J> \quad , \tag{6}$$

where $<f,g>:=\int\frac{d^4p}{(2\pi)^4}\,f(-p)g(p)$, $J\in S(\mathbf{R}^4)$.

The generating functional $\Gamma^{\alpha,\alpha_0}(\Phi_c)$ of the corresponding one-particle irreducible Green functions then can be obtained by a Legendre transformation:
Let $\delta_{J(p)}:=\delta/\delta J(p)$ and $\frac{1}{(2\pi)^4}\,\Phi_c(-p):=\delta_{J(p)}\,W_c^{\alpha,\alpha_0}(J)$. Then $\Gamma^{\alpha,\alpha_0}(\Phi_c)$ is defined by

$$\Gamma^{\alpha,\alpha_0}(\Phi_c):=\Big[\,W_c^{\alpha,\alpha_0}(J)-<J,\Phi_c>\Big]_{J=J(\Phi_c)} \quad , \tag{7}$$

implying

$$\delta_{\Phi_c(-p)}\,\Gamma^{\alpha,\alpha_0}(\Phi_c)\;=\;\frac{-1}{(2\pi)^4}\;J(p) \quad . \tag{8}$$

In order to compute $\Gamma^{\alpha,\alpha_0}(\Phi_c)$ we have to invert the equation

$$\Phi_c(p,J) = \tilde{C}_\alpha^{\alpha_0}(p) \Big\{ (2\pi)^4 \, \delta_{\Phi(-p)} \, L^{\alpha,\alpha_0}(\Phi) \, |_{\Phi=\tilde{C}_\alpha^{\alpha_0} J} - J(p) \Big\} \tag{9}$$

to get $J(p,\Phi_c)$. Because $L^{\alpha,\alpha_0}(\Phi) = \sum_{r=1}^{\infty} g^r L_r^{\alpha,\alpha_0}(\Phi)$ is a formal power series in g and $L_r^{\alpha,\alpha_0}(\Phi)$ is an (even due to the symmetry $\Phi \to -\Phi$) polynomial in Φ of degree $\leq 2r+2$ (connectedness), we can invert (9) up to any order r and therefore $\Gamma^{\alpha,\alpha_0}(\Phi_c)$ is well defined in the sense of a formal power series in g.

We have to keep in mind that now $\Phi_c(p)$ is viewed as the independent variable (as a function $\in S(\mathbf{R^4})$) and that $J(p,\Phi_c)$ is a formal power series in g, which depends on α and α_0.

3.2 The Flow Equation for $\Gamma^{\alpha,\alpha_0}(\Phi_c)$

By taking the derivative of (7) with respect to α we get

$$\partial_\alpha \Gamma^{\alpha,\alpha_0}(\Phi_c) \;=\; (\,\partial_\alpha W^{\alpha,\alpha_0})\,(J)\,|_{J=J(\Phi_c)} \quad . \tag{10}$$

We insert (6) and obtain

$$\partial_\alpha \Gamma^{\alpha,\alpha_0}(\Phi_c) = \Big[\partial_\alpha \, (\, L^{\alpha,\alpha_0}(\Phi) + I^{\alpha,\alpha_0}) \Big]_{\Phi=\tilde{C}_\alpha^{\alpha_0} J(\Phi_c)} \tag{11}$$

$$+ \Big[\int d^4q \, \Big(\partial_\alpha \, \tilde{C}_\alpha^{\alpha_0}(q) \Big) \, J(q) \, \delta_{\Phi(q)} \, L^{\alpha,\alpha_0}(\Phi) - \frac{1}{2} < J, (\partial_\alpha \, \tilde{C}_\alpha^{\alpha_0}) J > \Big]_{J=J(\Phi_c)} \quad .$$

Now we use the flow equation for $L^{\alpha,\alpha_0}(\Phi)$:

$$\partial_\alpha \, (\, L^{\alpha,\alpha_0}(\Phi) + I^{\alpha,\alpha_0}) = [\partial_\alpha \, \tilde{\Delta}(\alpha,\alpha_0)\,] \; L^{\alpha,\alpha_0}(\Phi) \tag{12}$$

$$-\frac{1}{2} < (2\pi)^4 \delta_\Phi \, L^{\alpha,\alpha_0}(\Phi), (\partial_\alpha \, \tilde{C}_\alpha^{\alpha_0}) \, (2\pi)^4 \delta_\Phi \, L^{\alpha,\alpha_0}(\Phi) > \quad ,$$

and get from (11), (12):

$$\partial_\alpha \Gamma^{\alpha,\alpha_0}(\Phi_c) \;=\; [\partial_\alpha \, \tilde{\Delta}(\alpha,\alpha_0)\,] \; L^{\alpha,\alpha_0}(\Phi) \, |_{\Phi=\tilde{C}_\alpha^{\alpha_0} J(\Phi_c)} \tag{13}$$

$$-\frac{1}{2} < (2\pi)^4 \delta_\Phi \, L^{\alpha,\alpha_0}(\Phi) - J(\Phi_c), (\partial_\alpha \, \tilde{C}_\alpha^{\alpha_0}) \Big((2\pi)^4 \delta_\Phi \, L^{\alpha,\alpha_0}(\Phi) - J(\Phi_c) \Big) > \, |_{\Phi=\tilde{C}_\alpha^{\alpha_0} J(\Phi_c)} \quad .$$

Together with (9) this yields

$$\partial_\alpha \, (\, \Gamma^{\alpha,\alpha_0}(\Phi_c) - \frac{1}{2} < \Phi_c, \{\tilde{C}_\alpha^{\alpha_0}\}^{-1} \Phi_c >) = [\partial_\alpha \, \tilde{\Delta}(\alpha,\alpha_0)\,] \; L^{\alpha,\alpha_0}(\Phi) \, |_{\Phi=\tilde{C}_\alpha^{\alpha_0} J(\Phi_c)} \quad . \tag{14}$$

In order to arrive at a differential flow equation for $\Gamma^{\alpha,\alpha_0}(\Phi_c)$ we have to express the functional on the right hand side of the equation (14) in terms of $\Gamma^{\alpha,\alpha_0}(\Phi_c)$.

We insert (9) into the relation

$$\delta(p+q) \;=\; \delta_{\Phi_c(p)}\,\Phi_c(-q) \tag{15}$$

and with the help of (8) we obtain

$$\delta(p+q) = -(2\pi)^8\int d^4q'\,\Big[\tilde{C}_\alpha^{\alpha_0}(q)\tilde{C}_\alpha^{\alpha_0}(q')\;\delta_{\Phi_c(p)}\delta_{\Phi_c(q')}\,\Gamma^{\alpha,\alpha_0}(\Phi_c) \tag{16}$$
$$\delta_{\Phi(-q')}\delta_{\Phi(q)}\,L^{\alpha,\alpha_0}(\Phi)\Big]_{\Phi=\tilde{C}_\alpha^{\alpha_0}J(\Phi_c)} + (2\pi)^4\,\tilde{C}_\alpha^{\alpha_0}(q)\,\delta_{\Phi_c(p)}\delta_{\Phi_c(q)}\,\Gamma^{\alpha,\alpha_0}(\Phi_c) \quad .$$

The right hand side of this equation is a formal power series in g. (16) has to be fulfilled up to any order r and therefore we get the equations:

$$\delta(p+q) = (2\pi)^4\,\tilde{C}_\alpha^{\alpha_0}(q)\;\delta_{\Phi_c(p)}\delta_{\Phi_c(q)}\,\Gamma_0^{\alpha,\alpha_0}(\Phi_c) \qquad \text{for}\;\; r=0 \quad , \tag{17}$$

$$\sum_{k=0}^{r-1} -(2\pi)^4\int d^4q'\;\tilde{C}_\alpha^{\alpha_0}(q)\tilde{C}_\alpha^{\alpha_0}(q')\;\hat{\Gamma}_{r-k}^{\alpha,\alpha_0}(q,-q',\Phi_c)\;\delta_{\Phi_c(p)}\delta_{\Phi_c(q')}\,\Gamma_k^{\alpha,\alpha_0}(\Phi_c) \tag{18}$$
$$+\,(2\pi)^4\,\tilde{C}_\alpha^{\alpha_0}(q)\;\delta_{\Phi_c(p)}\delta_{\Phi_c(q)}\,\Gamma_r^{\alpha,\alpha_0}(\Phi_c) = 0 \qquad ,\; r>0 \quad ,$$

with the definition

$$\hat{\Gamma}_r^{\alpha,\alpha_0}(q,p,\Phi_c) \;:=\; (2\pi)^4\Big\{\,\delta_{\Phi(p)}\delta_{\Phi(q)}\,L^{\alpha,\alpha_0}(\Phi)\,|_{\,\Phi=\tilde{C}_\alpha^{\alpha_0}J(\Phi_c)}\Big\}_r \quad . \tag{19}$$

We now can insert (17) into (18) and obtain

$$\hat{\Gamma}_r^{\alpha,\alpha_0}(q,p,\Phi_c) = (2\pi)^4\,\delta_{\Phi_c(p)}\delta_{\Phi_c(q)}\,\Gamma_r^{\alpha,\alpha_0}(\Phi_c) \tag{20}$$
$$-\,(2\pi)^4\sum_{k=1}^{r-1}\int d^4q'\;\tilde{C}_\alpha^{\alpha_0}(q')\;\hat{\Gamma}_{r-k}^{\alpha,\alpha_0}(q,-q',\Phi_c)\;\delta_{\Phi_c(p)}\delta_{\Phi_c(q')}\,\Gamma_k^{\alpha,\alpha_0}(\Phi_c) \quad .$$

(20) is a recursive relation for $\hat{\Gamma}_r^{\alpha,\alpha_0}(q,p,\Phi_c)$ and allows us to express $\hat{\Gamma}_r^{\alpha,\alpha_0}(q,p,\Phi_c)$ in terms of $\Gamma_k^{\alpha,\alpha_0}(\Phi_c)$, $k=1,\ldots,r$.

We compare the left and the right hand side of (14) in powers of g, use the definition (19) and end up with a differential flow equation for $\Gamma_r^{\alpha,\alpha_0}(\Phi_c)$:

$$\partial_\alpha\,\Gamma_r^{\alpha,\alpha_0}(\Phi_c) \;=\; \frac{1}{2}\int d^4p\;(\partial_\alpha\,\tilde{C}_\alpha^{\alpha_0}(p))\;\hat{\Gamma}_r^{\alpha,\alpha_0}(p,-p,\Phi_c) \quad , \quad r\geq 1 \;. \tag{21}$$

3.3 One-particle irreducible Green functions

$\Gamma_r^{\alpha,\alpha_0}(\Phi_c)$ is an even polynomial in Φ_c:

$$\Gamma_r^{\alpha,\alpha_0}(\Phi_c) = \sum_{n\geq 2} \int \prod_{k=1}^{n-1} \frac{d^4 p_k}{(2\pi)^4}\ \Phi_c(p_k)\, \Phi_c(-\sum_{j=1}^{n-1} p_j)\ \Gamma_{r,n}^{\alpha,\alpha_0}(p_1,\ldots,p_{n-1}) \quad . \tag{22}$$

$\Gamma_{r,n}^{\alpha,\alpha_0}$ is the momentum space regularized one-particle irreducible n-point Green function of order r.

It is defined to be symmetric under permutations of $p_1,\ldots,p_n := -\sum_{j=1}^{n-1} p_j$, it is invariant under $O(4)$-transformations of the p_j and it is in $C^\infty([\alpha_0,\infty)\times \mathbf{R}^{4(n-1)})$ as a function of α and $p_1,\ldots,p_{n-1}$. We have: $\Gamma_{r,n}^{\alpha,\alpha_0}\equiv 0$ for $n>2r+2$ (connectedness). (21) rewritten for the coefficient functions $\Gamma_{r,n}^{\alpha,\alpha_0}(p_1,\ldots,p_{n-1})$ yields:

$$\partial_\alpha \Gamma_{r,n}^{\alpha,\alpha_0}(p_1,\ldots,p_{n-1}) = \frac{1}{2}\int \frac{d^4 p}{(2\pi)^4}\ (\partial_\alpha \tilde{C}_\alpha^{\alpha_0}(p))\ \hat{\Gamma}_{r,n+2}^{\alpha,\alpha_0}(p,-p,p_1,\ldots,p_{n-1}) \quad , \tag{23}$$

where

$$\hat{\Gamma}_{r,n+2}^{\alpha,\alpha_0}(p,-p,p_1,\ldots,p_{n-1}) := (n+1)(n+2)\,\Gamma_{r,n+2}^{\alpha,\alpha_0}(p,-p,p_1,\ldots,p_{n-1}) \tag{24}$$

$$-\sum_{v=2}^{r}\sum_{\{a_j\},\{b_j\}} (-1)^v K^v(b_1,\ldots,b_v)\Big[\prod_{k=1}^{v-1} \tilde{C}_\alpha^{\alpha_0}(q'_k)\ \Gamma_{a_k,b_k+2}^{\alpha,\alpha_0}(q'_{k-1},p_{i_k+1},\ldots,p_{i_k+b_k})$$

$$\Gamma_{a_v,b_v+2}^{\alpha,\alpha_0}(q'_{v-1},-p,p_{i_v+1},\ldots,p_{n-1})\Big]_{\text{symm.}} \quad ,$$

with

$$q'_0 = p \quad , \quad q'_k = p + \sum_{j=1}^{b_1+\ldots+b_k} p_j \quad , \quad a_j>0 \quad , \quad b_j = 0,2,4,\ldots,n \quad , \quad i_k = \sum_{j=1}^{k-1} b_j \quad .$$

The sum is over all $\{a_j\}$ with $\sum_{j=1}^{v} a_j = r$ and over all $\{b_j\}$ with $\sum_{j=1}^{v} b_j = n$.
$K^v(b_1,\ldots,b_v)$ is a combinatorial factor, which could be computed with the help of (20) and $[\ldots]_{\text{symm.}}$ indicates the symmetrization operation with respect to $p_1,\ldots,p_n = -\sum_{j=1}^{n-1} p_j$.

We use (9) and (7) to compute the lowest order contribution in powers of $\tilde{C}_\alpha^{\alpha_0}$ to $\Gamma_r^{\alpha,\alpha_0}$ and as $\tilde{C}_{\alpha_0}^{\alpha_0}(p)\equiv 0$, we conclude that for $\alpha\to\alpha_0$ $\mathcal{L}_{r,n}^{\alpha_0,\alpha_0}\equiv \Gamma_{r,n}^{\alpha_0,\alpha_0}$ and therefore we get from (2) the boundary values at $\alpha=\alpha_0$:

$$\Gamma_{r,2}^{\alpha_0,\alpha_0}(p) \ = \ a_r^{\alpha_0} + b_r^{\alpha_0} p^2 \quad , \quad \Gamma_{r,4}^{\alpha_0,\alpha_0}(p_1,p_2,p_3) = c_r^{\alpha_0} \quad , \quad \Gamma_{r,n}^{\alpha_0,\alpha_0}(\vec{p})\equiv 0 \quad \text{for } n>4 \quad . \tag{25}$$

$\vec{p}$ denotes the tuple $(p_1,\ldots,p_{n-1})$.

(25) implies

$$\partial_p^\omega\, \Gamma_{r,n}^{\alpha_0,\alpha_0}(\vec{p}) \equiv 0 \quad \text{for} \quad n+|\omega|>4 \quad , \tag{26}$$

where $\omega = (\omega_1^0,\ldots,\omega_1^3,\omega_2^0,\ldots,\omega_{n-1}^3)$ is a multiindex and

$$\partial_p^{\omega} \equiv \partial_{p_1}^{\omega_1^0} \ldots \partial_{p_{n-1}}^{\omega_{n-1}^3} \qquad , \qquad |\omega| = \sum_{\mu=0}^{3} \sum_{j=1}^{n-1} \omega_j^{\mu} \quad .$$

With the help of (23), (24) and (25) all one-particle irreducible n-point Green functions of any order r can be computed by integrating successively (23) with respect to α from the lower bound α_0 up to the new parameter α, following the standard induction scheme of the flow equation method (see [3]) upwards in r and for given r downwards in n.

For illustration we can interpret the contributions to $\hat{\Gamma}_{r,n+2}^{\alpha,\alpha_0}$ in (24) in terms of Feynman graphs: We suppose $\Gamma_{r,n}^{\alpha,\alpha_0}$ can be written as a sum of one-particle irreducible Feynman graphs where every internal line is a function of the parameter α. If we now take the derivative of $\Gamma_{r,n}^{\alpha,\alpha_0}$ with respect to α, we can divide the expression obtained into contributions of two different types. Contributions of the first type stay one-particle irreducible, if we remove the line on which the derivative acts. The first term on the right hand side of (24) can be interpreted as a sum of all these contributions. Contributions of the second type become one-particle reducible, if we remove the line on which the derivative acts. The sum of these contributions corresponds to the second term on the right hand side of (24).

Due to these considerations we conclude that by integrating successively (23) we indeed get the one-particle irreducible Green functions (the proof could be carried out by induction in r and n).

4 Renormalizability and Analyticity of the one-particle irreducible Green functions

Let us now change the index pair (r,n) and take

$$l := r - \frac{n}{2} + 1 \quad \text{and} \quad s := 2r - \frac{n}{2} \tag{27}$$

as a new index pair to number our Green functions.(In the language of Feynman graphs l corresponds to the number of loops and s to the number of internal lines of an (unrenormalized) graph.) For our new indices (26) reads:

$$\partial_p^{\omega}\, \Gamma_{l,s}^{\alpha_0,\alpha_0}(\vec{p}) \equiv 0 \quad \text{for} \quad \frac{|\omega|}{2} + s > 2l \quad . \tag{28}$$

Furthermore we have:

$$\Gamma_{l,s}^{\alpha,\alpha_0}(\vec{p}) \equiv 0 \qquad \text{for} \quad \underbrace{s < 2l-1}_{n<2} \quad , \quad \underbrace{l < 0}_{n>2r+2} \quad , \quad \underbrace{s < 0}_{n>4r} \quad . \tag{29}$$

The flow equation (23) written for the new indices is

$$\partial_\alpha\, \Gamma_{l,s}^{\alpha,\alpha_0}(\vec{p}) = \frac{1}{2} \int \frac{d^4p}{(2\pi)^4}\, (\partial_\alpha\, \tilde{C}_\alpha^{\alpha_0}(p))\, \hat{\Gamma}_{l-1,s-1}^{\alpha,\alpha_0}(p,-p,\vec{p}) \quad , \tag{30}$$

where

$$\hat{\Gamma}^{\alpha,\,\alpha_0}_{l-1,s-1}(p,-p,\vec{p}) := h^s_l\;\Gamma^{\alpha,\,\alpha_0}_{l-1,s-1}(p,-p,\vec{p}) \tag{31}$$

$$-\sum_{v=2}^{s-l+1}\;\sum_{\{c_j\},\{d_j\}}(-1)^v\,K^v(b_1,\ldots,b_v)\Big[\prod_{k=1}^{v-1}\tilde{C}^{\alpha_0}_{\alpha}(q'_k)\;\Gamma^{\alpha,\,\alpha_0}_{c_k,d_k}(\vec{p}_k)\;\Gamma^{\alpha,\,\alpha_0}_{c_v,d_v}(\vec{p}_v)\Big]_{\text{symm.}}\quad .$$

The sum is over all $\{c_j\}$, $\{d_j\}$ with $\sum_{j=1}^{v} c_j = l-1$ and $\sum_{j=1}^{v} d_j + v = s$, where $c_j, d_j \geq 0$. Furthermore we have:

$$\vec{p}_k = (q'_{k-1}, p_{i_k+1},\ldots,p_{i_k+b_k})\quad,\quad b_k = 2d_k - 4c_k + 2\quad,\quad i_k = \sum_{j=1}^{k-1} b_j\quad,$$

$$q'_0 = p\quad,\quad q'_k = p + \sum_{j=1}^{b_1+\ldots+b_k} p_j\quad,\quad \vec{p}_v = (q'_{v-1}, p_{i_v+1},\ldots,p_{n-1},-p)\quad,\quad h^s_l = (n+2)(n+1)\;.$$

4.1 Integral Representation for $\partial^{\omega}_p\,\Gamma^{\alpha,\,\alpha_0}_{l,s}(\vec{p}\,)$

By successive integration of the flow equation (30) we now want to derive an integral representation for $\partial^{\omega}_p\,\Gamma^{\alpha,\,\alpha_0}_{l,s}$. Note that because we are in a massive theory we may integrate the flow equation (30) and its momentum derivatives up to infinity with respect to α since the mass provides an exponential infrared cutoff for large α.

We use the following boundary conditions (see also [3],[4]): At $\alpha = \alpha_0$ we impose (28). For the so-called relevant and marginal terms with $\frac{|\omega|}{2} + s \leq 2l$ we impose renormalization conditions at $\alpha = \infty$ by fixing the values of $\Gamma^{\infty,\,\alpha_0}_{l,2l}(0)$, $\Gamma^{\infty,\,\alpha_0}_{l,2l-1}(0)$ and $\partial_\mu\partial_\nu\,\Gamma^{\infty,\,\alpha_0}_{l,2l-1}(0)$ (We restrict to momentum 0 because we want to analyse the corresponding relativistic theory later on, and there it is convenient to start by renormalizing at 0 momentum). We have to distinguish three cases:

1. $s > 2l$

$$\Gamma^{\alpha,\,\alpha_0}_{l,s}(\vec{p}) = \int_{\alpha_0}^{\alpha} d\alpha'\;\partial_{\alpha'}\,\Gamma^{\alpha',\,\alpha_0}_{l,s}(\vec{p}) = \int_{\alpha_0}^{\alpha} d\alpha'\;(\text{r.h.s. of (30)})\quad . \tag{32}$$

2. $s = 2l$

$$\Gamma^{\alpha,\,\alpha_0}_{l,s}(\vec{p}) = \Gamma^{\alpha,\,\alpha_0}_{l,s}(0) + \sum_{j=1}^{3}\sum_{\mu=0}^{3} p_{\mu,j}\int_0^1 d\lambda\,\left(\partial_{\mu,j}\,\Gamma^{\alpha,\,\alpha_0}_{l,s}\right)(\lambda\,\vec{p})\quad , \tag{33}$$

with

$$\Gamma^{\alpha,\,\alpha_0}_{l,s}(0) = \Gamma^{\infty,\,\alpha_0}_{l,s}(0) - \int_{\alpha}^{\infty} d\alpha'\;\underbrace{\partial_{\alpha'}\,\Gamma^{\alpha',\,\alpha_0}_{l,s}(0)}_{\text{r.h.s. of (30)}} \tag{34}$$

and

$$\partial_{\mu,j}\,\Gamma_{l,s}^{\alpha,\,\alpha_0}(\vec{p}) = \int\limits_{\alpha_0}^{\alpha} d\alpha'\,\partial_{\mu,j}\,\underbrace{\partial_{\alpha'}\,\Gamma_{l,s}^{\alpha',\,\alpha_0}(\vec{p})}_{\text{r.h.s. of (30)}} \quad .$$

3. $s = 2l-1$

$$\Gamma_{l,s}^{\alpha,\,\alpha_0}(p) = \underbrace{\Gamma_{l,s}^{\alpha,\,\alpha_0}(0)}_{\text{as in (34)}} + \frac{1}{2}\sum_{\mu,\nu=0}^{3} p_\mu\,p_\nu\,\partial_\mu\,\partial_\nu\,\Gamma_{l,s}^{\alpha,\,\alpha_0}(0) \tag{35}$$

$$+ \sum_{\mu,\mu',\,\mu''=0}^{3} p_\mu \int\limits_0^1 d\lambda_1\,p_{\mu'}\lambda_1 \int\limits_0^1 d\lambda_2\,p_{\mu''}\lambda_2\lambda_1 \int\limits_0^1 d\lambda_3 \left(\partial_{\mu''}\partial_{\mu'}\partial_\mu\,\Gamma_{l,s}^{\alpha,\,\alpha_0}\right)(\lambda_3\lambda_2\lambda_1\,p) \quad ,$$

with

$$\partial_\mu\,\partial_\nu\,\Gamma_{l,s}^{\alpha,\,\alpha_0}(0) = \partial_\mu\,\partial_\nu\,\Gamma_{l,s}^{\infty,\,\alpha_0}(0) - \int\limits_{\alpha}^{\infty} d\alpha'\,\partial_\mu\,\partial_\nu\,\underbrace{\partial_{\alpha'}\,\Gamma_{l,s}^{\alpha',\,\alpha_0}(0)}_{\text{r.h.s. of (30)}}$$

and

$$\partial_{\mu''}\,\partial_{\mu'}\,\partial_\mu\,\Gamma_{l,s}^{\alpha,\,\alpha_0}(p) = \int\limits_{\alpha_0}^{\alpha} d\alpha'\;\partial_{\mu''}\,\partial_{\mu'}\,\partial_\mu\,\underbrace{\partial_{\alpha'}\,\Gamma_{l,s}^{\alpha',\,\alpha_0}(p)}_{\text{r.h.s. of (30)}} \quad .$$

For $l=0$ we get from (29), (30), (32) and (33):

$$\Gamma_{0,0}^{\alpha,\,\alpha_0}(\vec{p}) = \Gamma_{0,0}^{\infty,\,\alpha_0}(0) \quad \text{and} \quad \Gamma_{0,s}^{\alpha,\,\alpha_0}(\vec{p}) \equiv 0 \quad \text{for } s>0 \quad . \tag{36}$$

We impose for reasons of simplicity the renormalization conditions:

$$\Gamma_{0,0}^{\infty,\,\alpha_0}(0) := c_1^R \quad \text{and} \quad \Gamma_{l,2l}^{\infty,\,\alpha_0}(0) = 0\,,\; \Gamma_{l,2l-1}^{\infty,\,\alpha_0}(0) = 0\,,\; \partial_\mu\,\partial_\nu\,\Gamma_{l,2l-1}^{\infty,\,\alpha_0}(0) = 0 \quad \text{for} \quad l>0\,. \tag{37}$$

Note that once we have fixed the renormalization conditions the bare parameters appearing in (25) are determined uniquely [3],[4].

Using (30), (32), (33), (35) and the starting point (36), we obtain by induction an integral representation for $\partial_p^\omega\,\Gamma_{l,s}^{\alpha,\,\alpha_0}$:

Lemma 1.

$$\partial_p^\omega\,\Gamma_{l,s}^{\alpha,\,\alpha_0}(\vec{p}) = \int\limits_0^1 d\lambda_1 \ldots \int\limits_0^1 d\lambda_{\sigma(l,s)} \int\limits_{\alpha_0}^{\infty} d\alpha_1 \ldots \int\limits_{\alpha_0}^{\infty} d\alpha_s\;\partial_p^\omega\,G_{l,s}^{\alpha}(\vec{\alpha},\vec{\lambda},\vec{p}) \quad . \tag{38}$$

$\vec{\alpha}$ and $\vec{\lambda}$ denote the tuples $(\alpha_1,\ldots,\alpha_s)$ and $(\lambda_1,\ldots,\lambda_{\sigma(l,s)})$ and $\partial_p^\omega\,G_{l,s}^{\alpha}$ obeys the bounds

$$|\partial_p^\omega\,G_{l,s}^{\alpha}(\vec{\alpha},\vec{\lambda},\vec{p})| \;\leq\; e^{-m^2\sum_{j=1}^{s}\alpha_j}\,P(|\vec{p}|)\,Q(\sqrt{\alpha}_1,\ldots,\sqrt{\alpha}_s) \quad , \tag{39}$$

where P is a polynomial with nonnegative coefficients – independent of α – in $|p_{0,1}|,\ldots,|p_{3,n-1}|$ and Q is a nonnegative rational function which has no poles for $\alpha_i>0$.

Proof. The induction scheme is quite simple as on the right hand side of (30) there only appear contributions up to $(l-1, s-1)$ and therefore the induction could proceed for example in $l+s$. To give a short indication how the elementary proof works let us carry out the induction step for $s > 2l$, $|\omega| = 0$:

We employ the induction hypothesis (38) on the right hand side of (32) and interchange the loop-integration d^4p with the $d\lambda_1 \ldots d\lambda_{\sigma_{\max}} d\alpha_1 \ldots d\alpha_{s-1}$ integration which is justified because of (39). This yields (40), (41). Now we check that the new $G^{\alpha}_{l,s}$ obeys the bounds (39) and the induction step is completed. The induction step for $|\omega| \neq 0$ and $s = 2l$, $s = 2l-1$ is analogous. $\square$

$G^{\alpha}_{l,s}(\vec{\alpha}, \vec{\lambda}, \vec{p})$ and $\sigma(l,s)$ are determined through the following recursive relations:

1. $s > 2l$

$$G^{\alpha}_{l,s}(\vec{\alpha}, \vec{\lambda}, \vec{p}) = \tilde{G}_{l,s}(\vec{\alpha}, \vec{\lambda}', \vec{p})\, \Theta(\alpha - \alpha_s) \tag{40}$$

with

$$\tilde{G}_{l,s}(\vec{\alpha}, \vec{\lambda}', \vec{p}) := \frac{1}{2} \int \frac{d^4p}{(2\pi)^4}\; e^{-\alpha_s(p^2+m^2)} \Big\{ h^s_l\, G^{\alpha_s}_{l-1,s-1}(\vec{\alpha}, \vec{\lambda}, \vec{p}, p, -p) \tag{41}$$

$$- \sum_{v=2}^{s-l+1} \sum_{\{c_j\},\{d_j\}} (-1)^v K^v(b_1, \ldots, b_v) \Big[\prod_{k=1}^{v-1} e^{-\alpha_{w_k}(q'^2_k+m^2)}\, \Theta(\alpha_s - \alpha_{w_k})\, G^{\alpha_s}_{c_k,d_k}(\vec{\alpha}_k, \vec{\lambda}_k, \vec{p}_k)$$

$$G^{\alpha_s}_{c_v,d_v}(\vec{\alpha}_v, \vec{\lambda}_v, \vec{p}_v) \Big]_{\text{symm.}} \Big\} \quad .$$

The sum is over all $\{c_j\}$, $\{d_j\}$ with $\sum_{j=1}^{v} c_j = l-1$ and $\sum_{j=1}^{v} d_j + v = s$, where $c_j, d_j \geq 0$. Furthermore we have:

$$\vec{\alpha}_k = (\alpha_{f_k+1}, \ldots, \alpha_{f_k+d_k}) \quad , \quad \vec{\lambda}_k = (\lambda_{u_k+1}, \ldots, \lambda_{u_k+\sigma(c_k,d_k)}) \quad , \quad \vec{\lambda}' = (\lambda_1, \ldots, \lambda_{\tilde{\sigma}(l,s)}) \quad ,$$

$$w_k = \sum_{j=1}^{v} d_j + k \quad , \quad f_k = \sum_{j=1}^{k-1} d_j \quad , \quad u_k = \sum_{j=1}^{k-1} \sigma(c_j, d_j) \quad .$$

As we can add to (38) as many integrals $\int_0^1 d\lambda_j$ with new variables λ_j as we like without changing anything we take the maximum number of λ-integrals which appear on the right hand side of (30) during one induction step as our new number of λ-integrals. For $s > 2l$ we set

$$\sigma(l,s) := \tilde{\sigma}(l,s) := \max_v \{\sigma(l-1, s-1), n_v\} \quad \text{with} \quad n_v := \max_{\{c_j\},\{d_j\}} \{\sum_{k=1}^{v} \sigma(c_k, d_k)\} \ .$$

2. $s = 2l$

$$G^{\alpha}_{l,s}(\vec{\alpha}, \vec{\lambda}, \vec{p}) = -\tilde{G}_{l,s}(\vec{\alpha}, \vec{\lambda}', 0)\, \Theta(\alpha_s - \alpha) + \sum_{j=1}^{3} \sum_{\mu=0}^{3} p_{\mu,j}\, \Big(\partial_{\mu,j}\, \tilde{G}_{l,s}\Big)(\vec{\alpha}, \vec{\lambda}', \lambda_{\sigma(l,s)}\, \vec{p})\, \Theta(\alpha - \alpha_s) \quad . \tag{42}$$

Here we set $\sigma(l,s) := 1+\tilde\sigma(l,s)$.

3. $s = 2l-1$

$$G^{\alpha}_{l,s}(\vec\alpha,\vec\lambda,p) = \Big[-\tilde G_{l,s}(\vec\alpha,\vec\lambda',0) - \frac{1}{2}\sum_{\nu,\,\mu=0}^{3} p_\mu\, p_\nu\, \partial_\mu\partial_\nu \tilde G_{l,s}(\vec\alpha,\vec\lambda',0)\Big]\,\Theta(\alpha_s-\alpha) \tag{43}$$

$$+ \sum_{\mu,\,\mu',\,\mu''=0}^{3} \lambda^2_{\sigma(l,s)-2}\,\lambda_{\sigma(l,s)-1}\, p_\mu\, p_{\mu'}\, p_{\mu''}\, \Big(\partial_{\mu''}\partial_{\mu'}\partial_\mu\, \tilde G_{l,s}\Big)(\vec\alpha,\vec\lambda',\lambda_{\sigma(l,s)-2}\,\lambda_{\sigma(l,s)-1}\,\lambda_{\sigma(l,s)}\,p)\,\Theta(\alpha-\alpha_s)$$

$\sigma(l,s) := 3+\tilde\sigma(l,s)$.

Note that, as long as we keep $\alpha_0 > 0$ and the external momenta are real, we have absolute convergence of the integrals in (38), even for $\alpha = \infty$.

4.2 Convergence of the Integral Representation

In this section we want to examine the convergence of the integral representation of $\partial^w_p\, \Gamma^{\alpha,\,\alpha_0}_{l,s}$ as $\alpha_0 \to 0$ and $\alpha \to \infty$ in a complex domain obtained by continuing $p_{0,1}, p_{0,2}, \ldots, p_{0,n-1}$ to complex values (see (45) below).

From (40), (41), (42) and (43) we can realize inductively in $l+s$ that $\partial^w_p\, G^\alpha_{l,s}$ may be analytically continued in the zero components of the external momenta into any complex domain:

$$\partial^w_p\, G^\alpha_{l,s}(\vec\alpha,\vec\lambda,p_{0,1}+ik_{0,1},\underline{p}_1,\ldots,p_{0,n-1}+ik_{0,n-1},\underline{p}_{n-1}) \tag{44}$$

is well-defined for finite positive $\alpha,\vec\alpha$ and polynomially bounded in $\vec p$ for $k_{0,1},\ldots,k_{0,n-1}\in\mathbf{R}$. From now on we restrict the imaginary parts $\vec k_0 = (k_{0,1},\ldots,k_{0,n-1})$ to

$$|\sum_{j\in\tau_a} k_{0,j}\,| \le 2(m-\eta) \quad \text{for all} \quad \tau_a \subseteq \{1,\ldots,n\} \quad , \quad k_{0,n} = -\sum_{j=1}^{n-1} k_{0,j} \tag{45}$$

and want to show that we still can control the limit $\alpha\to\infty$ of $\partial^w_p\,\Gamma^{\alpha,\,\alpha_0}_{l,s}$. Here and in the following $\eta > 0$ is a fixed number which may be chosen arbitrarily small.

In this domain we can integrate $|\partial^w_p\, G^\alpha_{l,s}|$ with respect to $\vec\alpha$ and $\vec\lambda$ over the region indicated in (38) as long as we keep α finite. This could be shown by induction in $l+s$ using similar bounds as in (39):

$$|\partial^w_p\, G^\alpha_{l,s}(\vec\alpha,\vec\lambda,p_{0,1}+ik_{0,1},\underline{p}_1,\ldots,p_{0,n-1}+ik_{0,n-1},\underline{p}_{n-1})|$$
$$\le\; e^{s\alpha 4m^2 - m^2\sum_{j=1}^s \alpha_j}\, P(\,|\vec p|\,)\, Q(\sqrt{\alpha}_1,\ldots,\sqrt{\alpha}_s) \quad \text{for} \quad \vec k_0 \neq 0\ . \tag{46}$$

For $\vec k_0 = 0$ we can use the bounds (39).

Therefore

$$\partial_p^{\omega}\, \Gamma_{l,s}^{\alpha,\,\alpha_0}(p_{0,1}+ik_{0,1},\underline{p}_1,\ldots,p_{0,n-1}+ik_{0,n-1},\underline{p}_{n-1}) \tag{47}$$

is well-defined and the integrand is absolutely integrable. Moreover we are able to give bounds for the $|\partial_p^{\omega}\, \Gamma_{l,s}^{\alpha,\,\alpha_0}|$ which for large α do not depend on α or α_0 :

Proposition 2. *The integral representation of the $\partial_p^{\omega}\, \Gamma_{l,s}^{\alpha,\,\alpha_0}$ in the domain defined by (45) obeys the bounds*

$$\int_0^1 d\lambda_1 \ldots \int_0^1 d\lambda_{\sigma(l,s)} \int_{\alpha_0}^{\infty} d\alpha_1 \ldots \int_{\alpha_0}^{\infty} d\alpha_s \,|\, \partial_p^{\omega}\, G_{l,s}^{\alpha}(\vec{\alpha},\vec{\lambda},p_{0,1}+ik_{0,1},\underline{p}_1,\ldots,p_{0,n-1}+ik_{0,n-1},\underline{p}_{n-1})\,|$$

$$\leq \begin{cases} P_1(\,|\vec{p}|\,) & \text{for } \ \alpha \geq \hat{\alpha} \\ P_2(\,|\log(\alpha)|\,)\ P_3(\,|\vec{p}|\sqrt{\alpha}\,)\ \alpha^{\,-2l+s+\frac{|\omega|}{2}} & \text{for } \ \alpha_0 \leq \alpha \leq \hat{\alpha} \end{cases} \quad . \tag{48}$$

P_k are (each time they appear possibly new) polynomials with nonnegative coefficients which depend neither on α nor on α_0 , but on $\eta, l, s, \hat{\alpha}$. $\hat{\alpha} > \alpha_0$ is some finite fixed number (e.g.1).

Proof. The proof is performed by induction in $l+s$. We first consider the case $\alpha \leq \hat{\alpha}$ and $s > 2l$. Applying ∂_p^{ω} on the recursive relation (41), multiplying by $\Theta(\alpha-\alpha_s)$ and taking the sum of the absolute values of all contributions leads to an inequality both sides of which can be integrated with respect to $\vec{\alpha}$ and $\vec{\lambda}$ (due to the bounds (46)) over the domain indicated in (48). We change the order of integrations on the right hand side and employ the induction hypothesis for $\alpha_s \leq \hat{\alpha}$. Furthermore we bound the powers of the α_{w_k} on the right hand side by powers of α_s and the corresponding α_{w_k}-integrals

$$\int_{\alpha_0}^{\alpha_s} d\alpha_{w_k}\, e^{-\alpha_{w_k}(\,q_k'^2+m^2-(\sum_{j=1}^{b_1+\ldots+b_k} k_{0,j})^2\,)} \tag{49}$$

by $\alpha_s\cdot$ constant uniformly in $0 \leq \alpha_0 < \alpha_s \leq \hat{\alpha}$. We then obtain:

$$\text{l.h.s. of (48)} \ \leq \int_{\alpha_0}^{\alpha} d\alpha_s \int d^4p \Big\{ e^{-\alpha_s p^2}\, \alpha_s^{\,-2l+s+1+\frac{|\omega|}{2}}\, P_2(\,|\log(\alpha_s)|\,) \tag{50}$$

$$P_3(\,|\vec{p}|\sqrt{\alpha_s}, |p_0|\sqrt{\alpha_s},\ldots,|p_3|\sqrt{\alpha_s}\,)\Big\} \quad ,$$

where we have bounded appearing factors of the type $|k_{0,j}|\sqrt{\alpha_s}$ by $2(m-\eta)\sqrt{\hat{\alpha}}$. We now perform the loop-integration and get:

$$\text{l.h.s. of (48)} \ \leq \int_{\alpha_0}^{\alpha} d\alpha_s\, \alpha_s^{\,-2l+s-1+\frac{|\omega|}{2}}\, P_2(\,|\log(\alpha_s)|\,)\, P_3(\,|\vec{p}|\sqrt{\alpha_s}\,) \quad . \tag{51}$$

Estimating the integral on the right hand side yields the induction hypothesis and completes the induction step.

The induction step for $s = 2l$, $s = 2l-1$ and $\alpha \le \hat{\alpha}$ is also performed using (41), and therefore the argumentation is almost the same, but we have to split the contributions to (42), (43) with $\alpha_s \ge \alpha$ into two parts: The first part with $\alpha_s \le \hat{\alpha}$ can be treated as above. For the second part we employ the induction hypothesis for $\alpha_s > \hat{\alpha}$ on the right hand side of the inequality that we have obtained from (41). (Note that $\vec{p}, \vec{k}_0 = 0$ for these terms.) We bound the α_{w_k}-integrals (49) by

$$\int_0^\infty d\alpha_{w_k}\, e^{-\alpha_{w_k} m^2} \quad , \tag{52}$$

compute the loop-integral, estimate the α_s-integral and end up with a constant independent of α, α_0. Taking all contributions together we thus can reproduce the induction hypothesis. This completes the induction step for the case $\alpha \le \hat{\alpha}$ which we can call the renormalizability part of Proposition 2.

For the case $\alpha > \hat{\alpha}$ we first consider the induction step for $s > 2l$. Looking at (41) we examine the two types of contributions on the right hand side. The first type which has no reducible line is easy to handle since the loop-integration variables can be left real and therefore – after applying ∂_p^w, taking the absolute value, multiplying by $\Theta(\alpha - \alpha_s)$, integrating over the region indicated in (48) and employing the induction hypothesis for $\alpha_s \le \hat{\alpha}$ and for $\alpha \ge \alpha_s > \hat{\alpha}$ – we can reproduce the induction hypothesis for $\alpha > \hat{\alpha}$ without any difficulties. The second type which can be described graphically as a sum of chains that consist of one-particle irreducible Feynman graphs which are connected to their neighbours by a reducible line requires a more careful analysis, because the propagators corresponding to the reducible lines may increase exponentially in the α_{w_k}. Due to this fact we will have to add an imaginary part to the loop-integration variable p_0 that means instead of integrating along the real p_0-axis we want to integrate along the path $p_0 + ik_0$ with fixed k_0. (We are free to do so because (44) is polynomially bounded.) Let us now have a look at the exponent of a propagator corresponding to a reducible line. In order to get an exponential decrease we have to achieve

$$\Big[m^2 + (\underline{p} + \sum_{j=1}^{b_1+\ldots+b_k} \underline{p}_j)^2 + (p_0 + \sum_{j=1}^{b_1+\ldots+b_k} p_{0,j})^2 - (k_0 + \sum_{j=1}^{b_1+\ldots+b_k} k_{0,j})^2 \Big] \ge \varepsilon > 0 \, . \tag{53}$$

Furthermore the real part of the exponent of the differentiated propagator which corresponds to the line that closes the loop has to be negative to get an exponential decrease in α_s:

$$(m^2 + \underline{p}^2 + p_0^2 - k_0^2) \ge \varepsilon > 0 \tag{54}$$

and therefore we require

$$|k_0| \le m - \eta \quad . \tag{55}$$

We now want to show that for each fixed chain that means for each contribution to the second term of (41) with fixed $v, c_1, \ldots, c_v, d_1, \ldots, d_v$ and a fixed order of external momenta

$p_1, \ldots, p_{n-1}$ we can find an imaginary part k_0 for the loop-integration that fulfills (55) and (53) for all $k = 1, \ldots, v-1$. For a fixed chain we define

$$\hat{q}_k := \sum_{j=1}^{b_1+\ldots+b_k} k_{0,j} \quad , \quad k = 1, \ldots, v-1 \quad . \tag{56}$$

(45) implies

$$|\hat{q}_k| \leq 2(m-\eta) \quad \text{and} \quad |\hat{q}_k - \hat{q}_i| \leq 2(m-\eta) \quad \text{for all} \quad k, i \quad . \tag{57}$$

Now it is easy to realize

Lemma 3. *Let k_0 be a real number bounded by*

$$-\min\{0, \min_k\{\hat{q}_k\}\} - m + \eta \leq k_0 \leq -\max\{0, \max_k\{\hat{q}_k\}\} + m - \eta \quad , \tag{58}$$

then k_0 fulfills (55) and (53) for all k.

Proof. Because of (57) we can always find a k_0 which obeys (58), and from (58) we get

$$|\hat{q}_k + k_0| \ \leq \ m - \eta \quad \text{for all} \quad k \quad , \tag{59}$$

and therefore we can see that for this k_0 (55) and (53) are fulfilled for all k. □

Since we want to employ the induction hypothesis on the right hand side of (41) k_0 has to satisfy another condition:
$(k_0, k_{0,1}, \ldots, k_{0,b_1}), (k_0 + \hat{q}_1, k_{0,i_2+1}, \ldots, k_{0,i_2+b_2}), \ldots, (k_0 + \hat{q}_{v-1}, k_{0,i_v+1}, \ldots, k_{0,n-1}, -k_0)$ $(i_k = \sum_{j=1}^{k-1} b_j)$ have to be in the domain indicated by (45). Therefore we have to modify the bounds which k_0 has to obey. We define

$$\hat{m}_k := \max_{\tau_{a,k}}\{\hat{q}_{k-1} + \sum_{j \in \tau_{a,k}} k_{0,i_k+j}\} \quad , \quad \check{m}_k := \min_{\tau_{a,k}}\{\hat{q}_{k-1} + \sum_{j \in \tau_{a,k}} k_{0,i_k+j}\} \quad , \quad k = 1, \ldots, v \quad , \tag{60}$$

where

$$\hat{q}_0 := 0 \quad , \quad \tau_{a,k} \subseteq \{1, \ldots, b_k\} \quad .$$

Furthermore we define

$$\hat{m} := \max_k\{\hat{m}_k\} \quad \text{and} \quad \check{m} := \min_k\{\check{m}_k\} \quad . \tag{61}$$

(45) implies

$$\hat{m} \ \leq \ 2(m-\eta) \quad \text{and} \quad \check{m} \ \geq \ -2(m-\eta) \quad . \tag{62}$$

Now we are ready to prove

Lemma 4. *Let* k_0 *be a real number bounded by*

$$k_0 \;\leq\; \min\left\{ \begin{array}{l} 2(m-\eta) - \max\{0, \hat{m}\} \\ m - \eta - \max\{0, \max_k \{\hat{q}_k\}\} \end{array} \right. \tag{63}$$

$$k_0 \;\geq\; \max\left\{ \begin{array}{l} -2(m-\eta) - \min\{0, \check{m}\} \\ -m + \eta - \min\{0, \min_k \{\hat{q}_k\}\} \end{array} \right. \quad ,$$

then k_0 *fulfills (55), (53) and*

$$|k_0 + \hat{q}_{k-1} + \sum_{j \in \tau_{a,k}} k_{0,i_k+j}| \;\leq\; 2(m-\eta) \quad \textit{for all} \quad \tau_{a,k} \subseteq \{1, \ldots, b_k\} \quad , \quad k = 1, \ldots, v \quad . \tag{64}$$

Proof. Due to (45) we can always find a k_0 that obeys (63). It is not difficult to check that this k_0 – besides fulfilling (55), (53) due to Lemma 3 – also fulfills (64). □

Now we are able to carry out the induction step for $s > 2l$: For every chain on the right hand side of (41) we choose a corresponding k_0 which satisfies (55), (53) for all k and (64). Then we apply ∂_p^w, take the absolute value, multiply by $\Theta(\alpha - \alpha_s)$, integrate over the domain indicated in (48) and employ the induction hypothesis for $\alpha_s \leq \hat{\alpha}$ – for this contribution we can refer to the case $\alpha \leq \hat{\alpha}$ treated above – and for $\alpha \geq \alpha_s > \hat{\alpha}$ on the right hand side. We bound the α_{w_k}-integrals

$$\int_{\alpha_0}^{\alpha_s} d\alpha_{w_k}\, e^{-\alpha_{w_k}(\, q_k'^2 + m^2 - (k_0 + \hat{q}_k)^2)} \ldots$$

by

$$\int_0^\infty d\alpha_{w_k}\, e^{-\alpha_{w_k} \eta(2m-\eta)} \ldots$$

and end up with

$$\text{l.h.s of (48)} \;\leq\; P_1(\,|\vec{p}|\,) + \int_{\hat{\alpha}}^{\alpha} d\alpha_s \Big\{ \int d^4p\; e^{-\alpha_s(\, p^2 + m^2)}\, P_1(\,|\vec{p}|, |p_0|, \ldots, |p_3|\,) \tag{65}$$

$$+ \sum \int d^4p\; e^{-\alpha_s(\, p^2 - k_0^2 + m^2)}\, P_1(\,|\vec{p}|, |p_0|, \ldots, |p_3|\,) \Big\} \quad .$$

Now we perform the loop-integration, set $\alpha = \infty$ and since $m^2 - k_0^2 \geq \eta(2m - \eta) > 0$ we obtain the induction hypothesis for $\alpha > \hat{\alpha}$.

For (42) and (43) the argumentation is almost the same. Note in particular that $\lambda_j \in [0,1]$ for all $j = 1, \ldots, \sigma(l,s)$ so that if (45) is fulfilled for $\vec{k}_0$ it also holds for $\lambda_j \vec{k}_0$. This completes the proof of Proposition 2. □

Because α and α_0 only appear as bounds of the domain of integration in (38) and (48), we also get from Proposition 2 the convergence of the integral representation of $\partial_p^w\, \Gamma_{l,s}^{\alpha,\alpha_0}$ as $\alpha_0 \to 0$ and $\alpha \to \infty$. Thus we obtain

Theorem 5. *The one-particle irreducible renormalized Green functions of perturbative Euclidean massive* Φ_4^4

$$\Gamma_{l,s}^{\infty,0}(p_{0,1}+ik_{0,1},\underline{p}_1,\ldots,p_{0,n-1}+ik_{0,n-1},\underline{p}_{n-1}) \tag{66}$$

are analytic in $p_{0,1}+ik_{0,1},\ldots,p_{0,n-1}+ik_{0,n-1}$ *in the domain defined by*

$$(p_{0,1},\ldots,p_{0,n-1})\in\mathbf{R}^{n-1} \quad \textit{and} \quad (k_{0,1},\ldots,k_{0,n-1})\in\mathbf{R}^{n-1}$$

$$\textit{with} \qquad |\sum_{j\in\tau_a}k_{0,j}\,|<2m \quad \textit{for all} \quad \tau_a\subseteq\{1,\ldots,n\} \quad , \quad k_{0,n}=-\sum_{j=1}^{n-1}k_{0,j} \tag{67}$$

and smooth with respect to $(\underline{p}_1,\ldots,\underline{p}_{n-1})\in\mathbf{R}^{3(n-1)}$ *. The integrands of their integral representations (38) are absolutely integrable.*

5 Structure of the Integrands $G_{l,s}^{\alpha}$

Using (40), (41), (42) and (43) we now want to analyse the structure of the integrands $G_{l,s}^{\alpha}$. We can state

Proposition 6. *The integrands* $G_{l,s}^{\alpha}$ *of the renormalized one-particle irreducible Green functions* $\Gamma_{l,s}^{\alpha,0}$ *with the renormalization conditions (37) have the following structure:*

$$G_{l,s}^{\alpha}(\vec{\alpha},\vec{\lambda},\vec{p}) \;=\; \sum_j P_j(\vec{p})\,Q_j(\vec{\alpha},\vec{\lambda})\,e^{-\sum_{k,v=1}^{n-1}A_{kv}^j(\vec{\alpha},\vec{\lambda})\,p_kp_v}\,V_j^{\,\alpha}(\vec{\alpha})\,e^{-m^2\sum_{k=1}^{s}\alpha_k} \quad . \tag{68}$$

(a) $V_j^{\,\alpha}(\vec{\alpha})$ *are products of* Θ*-functions in* $(\alpha_i-\alpha_k)$, $\pm(\alpha-\alpha_s)$ *. The support of* $V_j^{\,\alpha}(\vec{\alpha})$ *restricts all* α_i *appearing as arguments of* $A_{kv}^j(\vec{\alpha},\vec{\lambda})$ *to* $\alpha_i\le\alpha$ *.*

(b) $A_{kv}^j(\vec{\alpha},\vec{\lambda})$ *are continuous with respect to* $\vec{\lambda}$ *and with respect to* $\vec{\alpha}$ *in the support of* $V_j^{\,\alpha}(\vec{\alpha})$ *. They are homogeneous of degree 1 in* $\vec{\alpha}$ *(that means* $A_{kv}^j(\tau\vec{\alpha},\vec{\lambda})=\tau\,A_{kv}^j(\vec{\alpha},\vec{\lambda})$ *), and* $A^j(\vec{\alpha},\vec{\lambda})$ *is a positive semi-definite symmetrical* $n-1\times n-1$ *matrix.*

(c) $Q_j(\vec{\alpha},\vec{\lambda})$ *are rational functions in* $\vec{\alpha}$ *and* $\vec{\lambda}$ *which are homogeneous of degree* $d_j\in\mathbf{Z}$ *in* $\vec{\alpha}$ *.*

(d) $P_j(\vec{p})=\prod_{k\le v}(p_kp_v)^{u_{k,v}^j}$, $u_{k,v}^j\in\mathbf{N}_0$ *, are monomials in* $O(4)$*-invariant scalar products of the* p_i *.*

All functions introduced on the right hand side of (68) also depend on l,s *and the sum over* j *is finite.*

Proof. The proof is carried out by induction in $l+s$. We employ the induction hypothesis on the right hand side of (41). First looking at the second term on the right hand side

we realize that this contribution can again be written as a sum of terms of the form (68). For these terms and also for the first term on the right hand side of (41) we thus obtain loop-integrals of the type

$$\int d^4p \; P_j(\vec{p}) \; e^{-\alpha_s p^2 - \sum_{k,v=1}^{n+1} A^j_{kv} \, p_k p_v} \quad , \tag{69}$$

where (according to (41)) $p_{n+1} = -p_n = -p$.

Using

$$- \alpha_s p^2 - \sum_{k,v=1}^{n+1} A^j_{kv} \, p_k p_v = - \sum_{k,v=1}^{n-1} \Big[A^j_{kv} - \frac{(A^j_{kn} - A^j_{kn+1})\,(A^j_{vn} - A^j_{vn+1})}{A^j_{nn} + A^j_{n+1n+1} - 2A^j_{nn+1} + \alpha_s} \Big] p_k p_v \tag{70}$$

$$-(A^j_{n+1n+1} + A^j_{nn} - 2A^j_{nn+1} + \alpha_s)\Big(p + \sum_{k=1}^{n-1} \frac{A^j_{kn} - A^j_{kn+1}}{A^j_{nn} + A^j_{n+1n+1} - 2A^j_{nn+1} + \alpha_s} \, p_k \Big)^2$$

we can perform the loop-integration and reproduce the induction hypothesis for $\tilde{G}_{l,s}$. Note that $\alpha_s \geq \alpha_i$ due to the induction hypothesis, see in particular (a). Therefore the second part in the square brackets in (70) is always well-defined. Furthermore the new contributions to the $P_j(\vec{p})$ are again of the form indicated in (d) which could be seen by using

$$(p\,p)^{\tilde{u}} \, (p\,p_k)^{\hat{u}} \, e^{p\,y} = (\nabla_y \nabla_y)^{\tilde{u}} \, (\nabla_y \, p_k)^{\hat{u}} \, e^{p\,y} \tag{71}$$

before computing the Gaussion integral and setting $y = 0$ afterwards. Inserting $\tilde{G}_{l,s}$ in (40), (42) and (43) again yields the induction hypothesis and completes the induction step. Note that the A^j's for 0-momentum $\tilde{G}$'s in (42), (43) are simply defined to be 0. □

In order to bound the degree of homogeneity d_j of $Q_j(\vec{\alpha}, \vec{\lambda})$ we define for fixed l, s

$$h_j \;:=\; 2 \sum_{k \leq v} u^j_{k,v} \quad . \tag{72}$$

Inserting (68) in (40), (41), (42) and (43) we can prove by induction in $l + s$ that the following equation holds:

$$\frac{1}{2} h_j \,-\, d_j \;=\; 2\,l \quad , \qquad \text{for all } \; l, s, j \quad . \tag{73}$$

We thus obtain for $G^{\infty}_{l,s}$ and for all l, s, j

$$d_j \;>\; -\,s \quad . \tag{74}$$

For $s > 2l$ this is obvious, and for $s = 2l$ we see in (42) that because $\alpha = \infty$ the first contribution is vanishing, and with the help of (68) we conclude that therefore only terms with $h_j \geq 2$ contribute. For $s = 2l - 1$ we get from (43) and (68) that for $\alpha = \infty$ only terms with $h_j \geq 4$ contribute.

Now it is easy to prove

Corollary 7. *The integral representation (38) of the one-particle irreducible renormalized Green functions* $\Gamma^{\infty,0}_{l,s}$ *with the renormalization conditions (37) can be written as*

$$\Gamma^{\infty,0}_{l,s}(\vec{p}) = \int_0^1 d\vec{\lambda} \int_0^1 d\beta_1 \ldots \int_0^1 d\beta_s\, \delta\Big(1-\sum_{k=1}^{s}\beta_k\Big)\Bigg[\sum_j V_j(\vec{\beta}\,)\, P_j(\vec{p}\,)\, Q_j(\vec{\beta},\vec{\lambda})$$

$$\frac{1}{\Big(\sum_{k,v} A^j_{kv}(\vec{\beta},\vec{\lambda})\, p_k p_v + m^2 \sum_{k=1}^{s}\beta_k\Big)^{d_j+s}}\Bigg] \quad , \tag{75}$$

and the integrand is absolutely integrable. The momentum derivatives $\partial^{\omega}_{p}\,\Gamma^{\infty,0}_{l,s}$ *are represented by integrals of the corresponding momentum derivatives of the integrand which are also absolutely integrable, and due to Theorem 5 this representation is still valid in the complex domain indicated in (67).*

Proof. Considering the integral representation (38) of the renormalized one-particle irreducible Green functions we define a substitution of the integration variables (we are free to do so because the integrand is absolutely integrable) by

$$\alpha_k =: \tau\beta_k \quad , \quad k = 1,\ldots,s \quad \text{and} \quad \sum_{k=1}^{s}\beta_k = 1 \quad . \tag{76}$$

This yields

$$d\alpha_1 \ldots d\alpha_s \;=\; \tau^{s-1}\,\delta\Big(1-\sum_{k=1}^{s}\beta_k\Big)\, d\beta_1 \ldots d\beta_s d\tau \quad . \tag{77}$$

Now we insert the integrand from Proposition 6 and get

$$\Gamma^{\infty,0}_{l,s}(\vec{p}) = \int_0^1 d\vec{\lambda} \int_0^1 d\beta_1 \ldots \int_0^1 d\beta_s \int_0^{\infty} d\tau\, \Bigg[\delta\Big(1-\sum_{k=1}^{s}\beta_k\Big)\, \sum_j V_j(\vec{\beta}\,)\, P_j(\vec{p}\,)$$

$$\tau^{s+d_j-1}\, Q_j(\vec{\beta},\vec{\lambda})\, e^{-\tau \sum_{k,v} A^j_{kv}(\vec{\beta},\vec{\lambda})\, p_k p_v}\; e^{-\tau\, m^2 \sum_{k=1}^{s}\beta_k}\Bigg] \quad . \tag{78}$$

Due to (74) and because the integrand is absolutely integrable we can perform the τ-integration and then we obtain (75). □

6 Relativistic massive Φ^4_4

6.1 The regularized Theory

We now want to turn our attention to the corresponding relativistic theory. We define a relativistic regularized propagator which is analytic in momentum space by

$$\tilde{C}_{\alpha}^{\alpha_0}(p) := \int_{\alpha_0}^{\alpha} d\alpha'\, e^{-\alpha'(\,p\eta p+(\varepsilon+i)m^2)} \quad , \qquad \varepsilon > 0\ ,\ 0 < \alpha_0 \leq \alpha < \infty \quad , \tag{79}$$

where η is the matrix

$$\eta := \begin{pmatrix} \varepsilon - i & 0 & 0 & 0 \\ 0 & \varepsilon + i & 0 & 0 \\ 0 & 0 & \varepsilon + i & 0 \\ 0 & 0 & 0 & \varepsilon + i \end{pmatrix} \qquad . \tag{80}$$

The interaction Lagrangian at scale α_0 is defined by

$$L^{\alpha_0,\,\alpha_0}(\Phi) := \sum_{r\geq 1} g^{\,r}\, L_r^{\alpha_0,\,\alpha_0}(\Phi)$$

and

$$L_r^{\alpha_0,\,\alpha_0}(\Phi) := \int d^4x\ \Big(a_r^{\alpha_0}\Phi^{\,2}(x) + b_r^{\alpha_0}\Phi(x)\Delta\Phi(x) - d_r^{\alpha_0}\Phi(x)\partial_0^2\Phi(x) + c_r^{\alpha_0}\Phi^{\,4}(x)\Big) \quad , \tag{81}$$

Δ denotes the 3-dim Laplace operator. As the ε-regularization breaks Lorentz invariance (but not $O(3)$- and T-invariance) the interaction Lagrangian contains an additional counterterm.

In analogy to the Euclidean theory the effective Lagrangian

$$L^{\alpha,\,\alpha_0}(\Phi) := \sum_{r\geq 1} g^{\,r}\, L_r^{\alpha,\,\alpha_0}(\Phi)$$

is defined through

$$e^{\,i\,L^{\alpha,\,\alpha_0}(\Phi)\ + i\,I^{\,\alpha,\,\alpha_0}} := e^{\Delta(\alpha,\alpha_0)}\ e^{\,i\,L^{\alpha_0,\,\alpha_0}(\Phi)} \quad , \tag{82}$$

where the functional Laplace operator $\Delta(\alpha, \alpha_0)$ is defined as in section 2 but with the relativistic propagator (79).

Differentiating (82) with respect to α yields the flow equation for the effective Lagrangian of the relativistic theory:

$$\partial_\alpha\, L^{\alpha,\,\alpha_0}(\Phi)\ +\ \partial_\alpha\, I^{\,\alpha,\,\alpha_0}\ =\ [\,\partial_\alpha\Delta(\alpha,\alpha_0)\,]\ L^{\alpha,\,\alpha_0}(\Phi) \tag{83}$$

$$+\frac{i}{2}\int d^4x \int d^4y\,\left(\frac{\delta}{\delta\,\Phi(x)}\,L^{\alpha,\,\alpha_0}(\Phi)\right)\left(\partial_\alpha\,C_\alpha^{\alpha_0}(x-y)\right)\frac{\delta}{\delta\,\Phi(y)}\,L^{\alpha,\,\alpha_0}(\Phi) \quad .$$

6.2 Flow Equations for one-particle irreducible Green functions

The generating functional $W_c^{\alpha,\alpha_0}(J)$ of the pertubative, regularized connected Green functions of the relativistic theory is given by

$$W_c^{\alpha,\alpha_0}(J) := i\,L^{\alpha,\alpha_0}(\Phi)\,|_{\,\Phi=i\tilde{C}_\alpha^{\alpha_0}J} + i\,I^{\alpha,\alpha_0} - \frac{1}{2} < J, \tilde{C}_\alpha^{\alpha_0} J > \quad . \tag{84}$$

Then the generating functional $i\,\Gamma^{\alpha,\alpha_0}(\Phi_c)$ of the corresponding one-particle irreducible Green functions is defined by

$$i\,\Gamma^{\alpha,\alpha_0}(\Phi_c) := \Big[\, W_c^{\alpha,\alpha_0}(J) - i < J, \Phi_c > \Big]_{\,J=J(\Phi_c)} \quad , \tag{85}$$

where

$$\Phi_c(p,J) = \frac{1}{i}(2\pi)^4 \delta_{J(-p)}\, W_c^{\alpha,\alpha_0}(J) = \tilde{C}_\alpha^{\alpha_0}(p) \Big\{ (2\pi)^4\, \delta_{\Phi(-p)}\, i\,L^{\alpha,\alpha_0}(\Phi)\,|_{\,\Phi=i\tilde{C}_\alpha^{\alpha_0}J} + i\,J(p) \Big\} \,. \tag{86}$$

Differentiating (85) with respect to α and using (84), (83) and (86) we get

$$\partial_\alpha \big(\, i\,\Gamma^{\alpha,\alpha_0}(\Phi_c) + \frac{1}{2} < \Phi_c, \{\tilde{C}_\alpha^{\alpha_0}\}^{-1} \Phi_c > \big) = [\partial_\alpha\, \tilde{\Delta}(\alpha,\alpha_0)\,]\; i\,L^{\alpha,\alpha_0}(\Phi)\,|_{\,\Phi=i\tilde{C}_\alpha^{\alpha_0}J(\Phi_c)} \quad . \tag{87}$$

We define

$$\hat{\Gamma}_r^{\alpha,\alpha_0}(q,p,\Phi_c) \;:=\; (2\pi)^4 \Big\{ \delta_{\Phi(p)}\delta_{\Phi(q)}\, L^{\alpha,\alpha_0}(\Phi)\,|_{\,\Phi=i\tilde{C}_\alpha^{\alpha_0}J(\Phi_c)} \Big\}_r \quad . \tag{88}$$

By the same procedure as in section 3.2 ((15),...,(20)) we now obtain a recursive relation for $\hat{\Gamma}_r^{\alpha,\alpha_0}(q,p,\Phi_c)$ that allows us to express $\hat{\Gamma}_r^{\alpha,\alpha_0}(q,p,\Phi_c)$ in terms of $\Gamma_k^{\alpha,\alpha_0}(\Phi_c)$, $k=1,\ldots,r$

$$\hat{\Gamma}_r^{\alpha,\alpha_0}(q,p,\Phi_c) = (2\pi)^4\, \delta_{\Phi_c(p)}\delta_{\Phi_c(q)}\, \Gamma_r^{\alpha,\alpha_0}(\Phi_c) \tag{89}$$
$$+\, i(2\pi)^4 \sum_{k=1}^{r-1} \int d^4q'\; \tilde{C}_\alpha^{\alpha_0}(q')\; \hat{\Gamma}_{r-k}^{\alpha,\alpha_0}(q,-q',\Phi_c)\; \delta_{\Phi_c(p)}\delta_{\Phi_c(q')}\, \Gamma_k^{\alpha,\alpha_0}(\Phi_c) \quad .$$

Using (87), (88) and (89) the differential flow equation for the relativistic one-particle irreducible Green functions $i\,\Gamma_{r,n}^{\alpha,\alpha_0}(p_1,\ldots,p_{n-1})$ reads:

$$\partial_\alpha\, i\,\Gamma_{r,n}^{\alpha,\alpha_0}(p_1,\ldots,p_{n-1}) = \frac{1}{2}\int \frac{d^4p}{(2\pi)^4}\; (\partial_\alpha\, \tilde{C}_\alpha^{\alpha_0}(p))\; i\,\hat{\Gamma}_{r,n+2}^{\alpha,\alpha_0}(p,-p,p_1,\ldots,p_{n-1}) \quad , \tag{90}$$

where

$$i\,\hat{\Gamma}_{r,n+2}^{\alpha,\alpha_0}(p,-p,p_1,\ldots,p_{n-1}) := (n+1)(n+2)\, i\,\Gamma_{r,n+2}^{\alpha,\alpha_0}(p,-p,p_1,\ldots,p_{n-1}) \tag{91}$$
$$+ \sum_{v=2}^{r} \sum_{\{a_j\},\{b_j\}} K^v(b_1,\ldots,b_v) \Big[\prod_{k=1}^{v-1} \tilde{C}_\alpha^{\alpha_0}(q'_k)\; i\,\Gamma_{a_k,b_k+2}^{\alpha,\alpha_0}(q'_{k-1},p_{i_k+1},\ldots,p_{i_k+b_k})$$

$$i\,\Gamma^{\alpha,\alpha_0}_{a_v,b_v+2}(q'_{v-1},-p,p_{i_v+1},\ldots,p_{n-1})\Big]_{\text{symm.}} \quad .$$

The notation is the same as before (see (24)).

Looking at (20), (23) and (24) we realize that multiplying the Euclidean one-particle irreducible Green functions by (-1) yields a flow equation which is – apart from the different definition of the propagators – identical to the flow equation for the relativistic one-particle irreducible Green functions.

Regarding the propagators we observe that we have to replace the Euclidean metric by the matrix η and that we have to add a factor $(\varepsilon+i)$ to the mass term m^2 in order to get the relativistic propagator. Therefore it turns out that as long as we keep $\varepsilon>0$ we can easily transfer most of our results from the Euclidean theory to the new situation.

6.3 Integral Representation and Renormalizability

The boundary values at $\alpha=\alpha_0$ follow from (81) and read:

$$\Gamma^{\alpha_0,\alpha_0}_{r,2}(p) \;=\; a^{\alpha_0}_r - b^{\alpha_0}_r \underline{p}^2 + d^{\alpha_0}_r p_0^2 \quad , \quad \Gamma^{\alpha_0,\alpha_0}_{r,4}(p_1,p_2,p_3)=c^{\alpha_0}_r \quad , \quad \Gamma^{\alpha_0,\alpha_0}_{r,n}(\vec{p})\equiv 0 \quad \text{for } n>4\,. \tag{92}$$

This implies (as in the Euclidean theory)

$$\partial^\omega_p\,\Gamma^{\alpha_0,\alpha_0}_{r,n}(\vec{p})\equiv 0 \quad \text{for} \quad n+|\omega|>4 \quad . \tag{93}$$

Changing the indices from (r,n) to (l,s) (see (27)) and using renormalization conditions which correspond to (37) multiplied by (-1)

$$i\,\Gamma^{\infty,\alpha_0}_{0,0}(0):=-c^R_1 \quad \text{and} \quad \Gamma^{\infty,\alpha_0}_{l,2l}(0)=0\,,\,\Gamma^{\infty,\alpha_0}_{l,2l-1}(0)=0\,,\,\partial_\mu\,\partial_\nu\,\Gamma^{\infty,\alpha_0}_{l,2l-1}(0)=0 \quad \text{for} \quad l>0\;, \tag{94}$$

we obtain in analogy to Lemma 1

Lemma 8.

$$\partial^\omega_p\, i\,\Gamma^{\alpha,\alpha_0}_{l,s}(\vec{p}) = \int_0^1 d\lambda_1\ldots\int_0^1 d\lambda_{\sigma(l,s)}\int_{\alpha_0}^\infty d\alpha_1\ldots\int_{\alpha_0}^\infty d\alpha_s\;\partial^\omega_p\,G^\alpha_{l,s}(\vec{\alpha},\vec{\lambda},\vec{p}) \quad , \tag{95}$$

and $\partial^\omega_p\,G^\alpha_{l,s}$ *obeys the bounds*

$$|\partial^\omega_p\,G^\alpha_{l,s}(\vec{\alpha},\vec{\lambda},\vec{p})| \;\le\; e^{-\varepsilon\, m^2\sum_{j=1}^s \alpha_j}\,P^\varepsilon(|\vec{p}|)\,Q(\sqrt{\alpha}_1,\ldots,\sqrt{\alpha}_s) \quad , \tag{96}$$

where P^ε *is a polynomial with nonnegative coefficients – independent of* α *– in* $|p_{0,1}|,\ldots,|p_{3,n-1}|$ *and* Q *is a nonnegative rational function which has no poles for* $\alpha_i>0$*.*

Proof. We refer to the proof of Lemma 1 in section 4.1. □

In order to examine the convergence of (95) as $\alpha_0 \to 0$ – keeping $\varepsilon > 0$ and the external momenta real – we now transfer our results of section 4.2 and state

Proposition 9. *The integral representation of the* $\partial_p^\omega \, i\,\Gamma_{l,s}^{\alpha,\alpha_0}$ *obeys the bounds*

$$\int_0^1 d\lambda_1 \ldots \int_0^1 d\lambda_{\sigma(l,s)} \int_{\alpha_0}^\infty d\alpha_1 \ldots \int_{\alpha_0}^\infty d\alpha_s \, |\, \partial_p^\omega \, G_{l,s}^\alpha(\vec\alpha, \vec\lambda, p_1, \ldots, p_{n-1})\, |$$

$$\leq \begin{cases} P_1^\varepsilon(\,|\vec p|\,) & \text{for } \ \alpha \geq \hat\alpha \\ P_2^\varepsilon(\,|\log(\alpha)|\,)\; P_3^\varepsilon(\,|\vec p|\sqrt{\alpha}\,)\; \alpha^{-2l+s+\frac{|\omega|}{2}} & \text{for } \ \alpha_0 \leq \alpha \leq \hat\alpha \quad . \end{cases} \tag{97}$$

P_k^ε *are (each time they appear possibly new) polynomials with nonnegative coefficients which depend neither on* α *nor on* α_0 *, but on* $\varepsilon, l, s, \hat\alpha$*.* $\hat\alpha > \alpha_0$ *is some finite fixed number (e.g.1).*

Proof. See the proof of Proposition 2. □

From Proposition 9 we directly get

Proposition 10. *The one-particle irreducible renormalized Green functions of perturbative relativistic massive* Φ_4^4 *with an* ε*-regularization given by (79) and (80)*

$$i\,\Gamma_{l,s}^{\infty,0}(p_1, \ldots, p_{n-1}) \tag{98}$$

are well-defined for $\varepsilon > 0$ *and smooth with respect to* $(p_1, \ldots, p_{n-1}) \in \mathbf{R}^{4(n-1)}$ *. The integrands of their integral representations (95) are absolutely integrable.*

6.4 The Limit $\varepsilon \to 0$

In order to analyse the limit $\varepsilon \to 0$ we need more information about the structure of the integrands $G_{l,s}^\alpha$. Using our results of section 5 we can state

Proposition 11. *The integrands* $G_{l,s}^\alpha$ *of the renormalized one-particle irreducible relativistic Green functions* $i\,\Gamma_{l,s}^{\alpha,0}$ *with an* ε*-regularization given by (79), (80) and the renormalization conditions (94) have the following structure:*

$$G_{l,s}^\alpha(\vec\alpha, \vec\lambda, \vec p) \;=\; \sum_j P_j^\varepsilon(\,\vec p\,)\, Q_j(\vec\alpha, \vec\lambda)\, e^{-\sum_{k,v} A_{kv}^j(\vec\alpha,\vec\lambda)\, p_k \eta p_v}\, V_j^{\ \alpha}(\vec\alpha) \tag{99}$$
$$(\varepsilon - i)^{-\frac{1}{2}l}(\varepsilon + i)^{-\frac{3}{2}l}\, e^{-(\varepsilon+i)m^2 \sum_{k=1}^s \alpha_k}$$

(a) $A_{kv}^j(\vec\alpha, \vec\lambda)$, $Q_j(\vec\alpha, \vec\lambda)$ *and* $V_j^{\ \alpha}(\vec\alpha)$ *are identical to the corresponding functions in the Euclidean theory.*

(b) $P_j^\varepsilon(\vec{p}) = \prod_{k\leq v} (p_k\,\eta\, p_v)^{u^j_{k,v}}$ *(with the same* $u^j_{k,v}$ *as in the Euclidean theory) are monomials, which for* $\varepsilon = 0$ *are invariant under Lorentz transformations.*

Proof. See the proof of Proposition 6. □

In analogy to Corollary 7 we can easily prove

Corollary 12. *The integral representation (95) of the one-particle irreducible renormalized relativistic Green functions* $i\,\Gamma^{\infty,0}_{l,s}$ *with an* ε*-regularization ((79), (80)) and the renormalization conditions (94) can be written as*

$$i\,\Gamma^{\infty,0}_{l,s}(\,\vec{p}\,) \;=\; \int_0^1 d\vec{\lambda} \int_0^1 d\beta_1 \ldots \int_0^1 d\beta_s\, \delta\Big(1 - \sum_{k=1}^{s} \beta_k\Big) \Bigg[\sum_j V_j(\vec{\beta}\,)\, P_j^\varepsilon(\,\vec{p}\,)\, Q_j(\vec{\beta},\vec{\lambda}) \\ \frac{(\varepsilon - i)^{-\frac{1}{2}l}\,(\varepsilon + i)^{-\frac{3}{2}l}\, i^{\,d_j+s}}{\Big(\sum_{k,v} A^j_{kv}(\vec{\beta},\vec{\lambda})\, i\, p_k \eta p_v + \,(i\varepsilon - 1)\, m^2 \sum_{k=1}^{s} \beta_k\Big)^{d_j+s}} \Bigg] \quad . \tag{100}$$

The integrand is absolutely integrable. The momentum derivatives $\partial_p^\omega\, i\,\Gamma^{\infty,0}_{l,s}$ *are represented by integrals of the corresponding momentum derivatives of the integrand which are also absolutely integrable.*

Proof. See the proof of Corollary 7. □

Let $\Psi(\vec{p}) \in S(\mathbf{R}^{4(n-1)})$ and

$$F_j^\varepsilon(\Psi,\vec{\beta},\vec{\lambda}) = \int d^4\vec{p}\;\Psi(\vec{p})\, \frac{P_j^\varepsilon(\vec{p})\,(\varepsilon - i)^{-\frac{1}{2}l}\,(\varepsilon + i)^{-\frac{3}{2}l}}{\Big(\sum_{k,v} A^j_{kv}(\vec{\beta},\vec{\lambda})\, i\, p_k \eta p_v + \,(i\varepsilon - 1)\, m^2 \sum_{k=1}^{s} \beta_k\Big)^{d_j+s}} \quad . \tag{101}$$

Looking at the denominator in (101) we realize that it has the structure

$$\Big(\mathrm{P}_1 \,+\, i\,\varepsilon\,\mathrm{P}_2 \,-\, m^2 + i\,\varepsilon\, m^2\Big)^{z} \tag{102}$$

with

$$\mathrm{P}_1 \;=\; \sum_{k,v} A^j_{kv}(\vec{\beta},\vec{\lambda})(p_{0,k}p_{0,v} - \underline{p}_k\,\underline{p}_v) \tag{103}$$

and

$$\mathrm{P}_2 \;=\; \sum_{k,v} A^j_{kv}(\vec{\beta},\vec{\lambda})(p_{0,k}p_{0,v} + \underline{p}_k\,\underline{p}_v) \quad . \tag{104}$$

Because $A^j(\vec{\beta},\vec{\lambda})$ is positive semi-definite and continuous in $\vec{\beta},\vec{\lambda}$ in the compact region of integration we can apply a theorem due to Speer (p.105 [22]) that tells us that for $\varepsilon \to 0$ (102) defines a tempered distribution which depends continuously on $A^j(\vec{\beta},\vec{\lambda})$ and that (102) has the same distributional limit as

$$\Big(\mathrm{P}_1 \,-\, m^2 + i\,\varepsilon\, m^2\Big)^{z} \tag{105}$$

which is Lorentz invariant. Therefore we conclude that for $\varepsilon \to 0$ (101) defines a Lorentz invariant tempered distribution which depends continuously on $A^j(\vec{\beta}, \vec{\lambda})$.

In the compact region of integration

$$\sum_j V_j(\vec{\beta}\,)\, Q_j(\vec{\beta}, \vec{\lambda})\, F_j^{\varepsilon}(\Psi, \vec{\beta}, \vec{\lambda}) \tag{106}$$

is absolutely integrable for all $\varepsilon > 0$ and $F_j^0(\Psi, \vec{\beta}, \vec{\lambda})$ is continuous. Therefore we conclude that (106) is still absolutely integrable for $\varepsilon = 0$.

Thus we can state

Theorem 13. *The limit $\varepsilon \to 0$ of the one-particle irreducible renormalized Green functions of perturbative relativistic massive Φ_4^4 defined by (79), (80), (81) and (94)*

$$\lim_{\varepsilon \to 0}\, i\, \Gamma_{l,s}^{\infty,0}(p_1, \ldots, p_{n-1}) \tag{107}$$

exists as a Lorentz invariant tempered distribution $\in S'(\mathbf{R}^{4(n-1)})$.

Remark. Using the flow equation (83) written for the perturbative, regularized amputated connected Green functions $i\mathcal{L}_{l,s}^{\alpha,\alpha_0}(\vec{p})$ a theorem analogous to Theorem 13 for these Green functions could be proved by a similar line of argumentation.

Comparing the corresponding integrands of the integral representations in the (multiplied by (-1)) Euclidean theory (75) and in the relativistic theory (100) we realize that they coincide as functions up to a factor $(-i)^{s-l}$ if in (75) the imaginary parts $k_{0,v}$ of the zero components of the external momenta take values in the domain (45) and the real parts are equal to 0, and if we let $\varepsilon \to 0$ and set $p_{0,v} = k_{0,v}$ in (100).

According to Theorem 5 these integrands are absolutely integrable and therefore we can conclude

Theorem 14. *The limit $\varepsilon \to 0$ of the one-particle irreducible renormalized Green functions of perturbative relativistic massive Φ_4^4 defined by (79), (80), (81) and (94)*

$$\lim_{\varepsilon \to 0}\, i\, \Gamma_{l,s}^{\infty,0}(p_1, \ldots, p_{n-1}) \tag{108}$$

exist as Lorentz invariant smooth functions in the domain D' defined as follows:

$$D := \Big\{ (p_{0,1}, \ldots, p_{0,n-1}) \in \mathbf{R}^{n-1} \textit{ with } \big| \sum_{j \in \tau_a} p_{0,j} \big| < 2m \quad \textit{for all} \quad \tau_a \subseteq \{1, \ldots, n\}\ ,$$

$p_{0,n} = -\sum_{j=1}^{n-1} p_{0,j}$ *and* $(\underline{p}_1, \ldots, \underline{p}_{n-1}) \in \mathbf{R}^{3(n-1)} \Big\}$; $D' := \bigcup_{\Lambda \in L} \Lambda D$, $L \cong$ *Lorentz group.*

References

[1] K.Wilson, J.B.Kogut: Phys.Rep.12C,75 (1975).

[2] J. Polchinski: Renormalization and Effective Lagrangians. Nucl.Phys.B231,269-295 (1984).

[3] G.Keller, Ch.Kopper, M.Salmhofer: Perturbative Renormalization and Effective Lagrangians in Φ_4^4. Helv.Phys.Acta 65,32-52 (1991).

[4] G.Keller, Ch.Kopper: Perturbative Renormalization of Composite Operators via Flow Equations I. Commun.Math.Phys.148,445-467 (1992).

[5] G.Keller, Ch.Kopper: Perturbative Renormalization of Composite Operators via Flow Equations II: Short distance expansion. Commun.Math.Phys.153,245-276 (1993).

[6] G.Keller: The Perturbative Construction of Symanzik's improved Action for Φ_4^4 and QED_4. Helv.Phys.Acta 66,453 (1993).

[7] Ch.Kopper, V.A.Smirnov: Analytic Regularization and Minimal Subtraction for Φ_4^4 with Flow Equations. Z.Phys.C 59,641-645 (1993).

[8] G.Keller: Local Borel summability of Euclidean Φ_4^4: A simple Proof via Differential Flow Equations. Commun.Math.Phys.161,311-323 (1994).

[9] G.Keller, Ch.Kopper: Perturbative Renormalization of QED via Flow Equations. Phys.Lett.B273,323-332 (1991).

[10] G.Keller, Ch.Kopper: Perturbative Renormalization of Massless Φ_4^4 with Flow Equations. Commun.Math.Phys.161,515-532 (1994).

[11] G.Keller, Ch.Kopper: Renormalizability Proof for QED based on Flow Equations. Commun.Math.Phys.176,193-226 (1996).

[12] H.Epstein, V.Glaser, R.Stora: General Properties of the n-Point Functions in Local Quantum Field Theory. Les Houches 1975; R.Balian, D.Iagolnitzer (eds.); North-Holland (1976).

[13] O.Steinmann: Perturbation Expansions in Axiomatic Field Theory. Lecture Notes in Physics 11, Springer 1971.

[14] H.Epstein, V.Glaser: The Role of Locality in Perturbation Theory. Ann.Inst.H.Poincaré 19,211-295 (1973).

[15] H.Epstein, V.Glaser: Adiabatic Limit in Perturbation Theory. Erice Advanced Study Institute 1975; G.Velo, A.Wightman (eds.); D.Reidel Publishing Company (1976).

[16] K.Hepp: Renormalization Theory. Statistical Mechanics and Quantum Field Theory, pp.429-500; C.DeWitt, R.Stora (eds.); Gordon and Breach (1971).

[17] R.J.Eden, P.V.Landshoff, D.I.Olive, J.C. Polkinghorne: The Analytic S-Matrix. Cambridge Univ. Press 1966.

[18] K.Hepp: Proof of the Bogoliubov-Parasiuk Theorem on Renormalization. Commun.Math.Phys.2,301-326 (1966).

[19] K.Hepp: Théorie de la renormalisation. Lecture Notes in Physics 2, Springer 1969.

[20] C.Chandler: Some Physical Region Mass Shell Properties of Renormalized Feynman Integrals. Commun.Math.Phys.19,169-188 (1970).

[21] V. Rivasseau: private Communication

[22] E.Speer: Generalized Feynman Amplitudes. Annals of Math. Studies No.62, Princeton Univ. Press 1969.